

Superconformal Anomaly from AdS/CFT Correspondence [†]

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Abstract

For a classical superconformal gauge theory in a conformal supergravity background, its chiral R-symmetry anomaly, Weyl anomaly and super-Weyl anomaly constitute a supermultiplet. We review how these anomalies arise from the five-dimensional gauged supergravity in terms of the AdS/CFT correspondence at the gravity level. The holographic production of this full superconformal anomaly multiplet provides a support and test to the celebrated AdS/CFT conjecture.

1 Introduction

The discovery of a D -brane as a fundamental dynamical object carrying $R-R$ charge has played a crucial role in establishing a web of dualities among five superstring theories and unifying them into a single M-theory [1]. On one hand, combining other types of branes such as the Neveu-Schwarz (NS) solitonic five-brane and orientifold plane, a supersymmetric gauge theory defined on the world volume of Dp -branes can be constructed from an elaborated setting of a brane configuration in a weakly type II string theory [2]. On the other hand, a stack of Dp -branes can modify the space-time background of the strong coupled type II string theory and arise as the brane solution to the low-energy effective theory of type II string, i.e., type II (A or B) supergravity. These two distinct features of a D -brane at both strong and weakly superstring theory have led to the celebrated AdS/CFT correspondence conjecture proposed by Maldacena [3].

The original AdS/CFT correspondence conjecture [3] states that the type IIB string theory compactified on $AdS_5 \times S^5$ theory with N units of $R-R$ flux on S^5 describes the same physics as $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills theory. The explicit definition was further clarified and generalized as the following [4, 5]. Given the type II superstring theory in the background $AdS_{d+1} \times X^{9-d}$, with X^{9-d} being a compact Einstein manifold, there should exist a one-to-one correspondence between a string state a supergravity field in the AdS_{d+1} bulk and a gauge invariant operator of the conformal field theory defined on the boundary of AdS_{d+1} space-time. Concretely, the partition function $Z_{\text{String}}[\phi_0]$ of the type II superstring theory as a functional of the boundary value $\phi_0(x)$ of a bulk field $\phi(x, r)$ should equal the generating functional of the correlation function Z_{CFT} of the gauge invariant operator of the conformal conformal field theory with the external source $\phi_0(x)$ provided by the boundary value of the bulk field $\phi(x, r)$,

$$\begin{aligned} Z_{\text{String}}[\phi_0] &= Z_{\text{CFT}}[\phi_0], \\ Z_{\text{String}}[\phi_0] &= \int_{\phi(x,0)=\phi_0(x)} \mathcal{D}\phi(x, r) \exp(-S[\phi(x, r)]), \\ Z_{\text{CFT}}[\phi_0] &= \left\langle \exp \int_{M^d} d^d x \mathcal{O}(x) \phi_0(x) \right\rangle \end{aligned}$$

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$$\begin{aligned}
&= \sum_n \frac{1}{n!} \int \prod_{i=1}^n d^d x_i \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \phi_0(x_1) \cdots \phi_0(x_n) \\
&\equiv \exp(-\Gamma_{\text{CFT}}[\phi_0]) .
\end{aligned} \tag{1}$$

In above equation, \mathcal{O} represent certain composite operator in the superconformal field theory such as the energy-momentum tensor and chiral R -symmetry current etc., ϕ_0 are the corresponding background fields such as the gravitational and gauge fields etc. coming from the boundary value of the corresponding bulk field, and $\Gamma_{\text{CFT}}[\phi_0]$ is the quantum effective action describing the composite operators interacting with background field ϕ_0 .

At low-energy the string effect can be neglected. The partition function of the type IIB superstring can be evaluated as the exponential of the type IIB supergravity action in a on-shell field configuration $\phi^{\text{cl}}[\phi^0]$ with the boundary value $\phi_0(x)$, i.e.,

$$Z_{\text{String}}[\phi_0] = \exp\left(-S_{\text{SUGRA}}[\phi^{\text{cl}}[\phi_0]]\right) . \tag{2}$$

A comparison between Eq. (1) and Eqs. (2) immediately shows that the large- N quantum effective action of the d -dimensional conformal field theory in the background provided by ϕ_0 can be approximately equal to the on-shell classical action of AdS_{d+1} supergravity with non-empty boundary,

$$\Gamma_{\text{CFT}}[\phi_0] = S_{\text{SUGRA}}[\phi^{\text{cl}}[\phi_0]] = \int d^d x \phi_0(x) \langle \mathcal{O} \rangle . \tag{3}$$

Let us see how the five-dimensional gauged supergravities [6, 7, 8] arises in the AdS/CFT correspondence [9]. The $AdS_5 \times S^5$ background comes from the near horizon limit of three-brane solution of type IIB supergravity [10]. Therefore, in the $AdS_5 \times S^5$ background, the spontaneous compactification on S^5 of the type IIB supergravity occurs [11]. With the assumption that there exists a consistent nonlinear truncation of the massless modes from the whole Kluza-Klein spectrum of the type IIB supergravity compactified on S^5 [12, 13], the resultant theory should be the $SO(6)(\cong SU(4))$ gauged $\mathcal{N} = 8$ AdS_5 supergravity since the isometry group $SO(6)$ of S^5 becomes the gauge group of the compactified theory and the $AdS_5 \times S^5$ background preserves all the supersymmetries of type IIB supergravity [8]. Furthermore, if the background for the type IIB supergravity is $AdS_5 \times X^5$ with X^5 being an Einstein manifold rather than S^5 such as $T^{1,1} = (SU(2) \times SU(2))/U(1)$ or certain orbifold, then the number of preserved supersymmetries in the compactified AdS_5 supergravity is reduced [14, 15]. One can thus obtain the gauged $\mathcal{N} = 2, 4$ AdS_5 supergravity in five dimensions, and their dual field theories should be $\mathcal{N} = 1, 2$ supersymmetric gauge theories [9]. A supersymmetric gauge theory with lower supersymmetries is not a conformal invariant theory since its beta function usually does not vanish. However, it was shown that the beta function a supersymmetric gauge theory has the zero point, at which the (super)conformal invariance can arise [16, 17]. The AdS/CFT correspondence between the $\mathcal{N} = 2, 4$ gauged supergravities in five dimensions and $\mathcal{N} = 1, 2$ supersymmetric gauge theories can thus be established [9].

Eq. (3) shows clearly that according the AdS/CFT correspondence a quantum effective action describing a superconformal gauge theory in an external supergravity background can be identified with the on-shell action of the gauged supergravity with non-trivial boundary data. Thus the various anomalies at the leading-order of large- N expansion of a four-dimensional supersymmetric gauge theory should be extracted from the AdS_5 gauged supergravity. Specifically, the AdS/CFT correspondence exchange strong and weak couplings and vice versa, and the

anomalies are independent to the couplings, the production of the superconformal anomaly from the gauged supergravity will provide a support and test to the *AdS/CFT* correspondence at the supergravity level.

This short review is outlined as the following. In Sect. II, we shall briefly introduce a classical superconformal gauge theory in a conformal supergravity background and explain how a superconformal anomaly multiplet arises. In Sect. III we shall explain the structure of the gauged supergravity in five dimensions and emphasize why the superconformal anomaly multiplet can be reproduced from the gauged supergravity in terms of the *AdS/CFT* correspondence. Sect. IV contains a systematic review on how chiral-, Weyl- and super-Weyl anomalies arise from the classical five-dimensional gauged supergravity. Sect. V is a brief summary.

2 Superconformal Anomaly Multiplet in External Conformal Supergravity Background

We focus on a general $\mathcal{N} = 1$ four-dimensional supersymmetric $SU(N)$ gauge theory, its conserved currents including the energy-momentum tensor $\theta^{\mu\nu}$, the supersymmetry current s^μ and the chiral (or equivalently axial vector) R -current j^μ constitute a supermultiplet due to the supersymmetry [18, 19],

$$\partial_\mu T^{\mu\nu} = \partial_\mu s^\mu = \partial_\mu j^\mu = 0. \quad (4)$$

If these currents satisfy further algebraic constraints, $T^\mu_\mu = \gamma_\mu s^\mu = 0$, the Poincaré supersymmetry will be promoted to a superconformal symmetry since one can construct three more conserved currents,

$$d^\mu \equiv x_\nu T^{\nu\mu}, \quad k_{\mu\nu} \equiv 2x_\nu x^\rho T_{\rho\mu} - x^2 T_{\mu\nu}, \quad l_\mu \equiv i x^\nu \gamma_\nu s_\mu. \quad (5)$$

These three new conserved currents lead to the generators for dilatation, conformal boost and conformal supersymmetry transformation. However, the superconformal symmetry may become anomalous at quantum level. If all of them, the trace of energy-momentum tensor, T^μ_μ , the γ -trace of supersymmetry current, $\gamma^\mu s_\mu$ and the divergence of the chiral R -current, $\partial_\mu j^\mu$, get contribution from quantum effects, they will form a chiral supermultiplet with the $\partial_\mu j^\mu$ playing the role of the lowest component of the corresponding composite chiral superfield [19, 20, 21].

When considering the $\mathcal{N} = 1$ supersymmetric gauge theory in a $\mathcal{N} = 1$ conformal supergravity background, the energy-momentum tensor $T_{\mu\nu}$, the supersymmetry current s_μ , and the chiral (or axial vector) R -symmetry current j_μ will couple to the gravitational field $g_{\mu\nu}$, chiral (or axial) vector field A_μ and vector-spinor gravitino field ψ_μ in the multiplet of conformal supergravity, respectively [22],

$$\mathcal{L}_{\text{ext}} = \int d^4x \sqrt{-g} \left(g_{\mu\nu} T^{\mu\nu} + A_\mu j^\mu + \bar{\psi}_\mu s^\mu \right). \quad (6)$$

The action (6) shows that the covariant conservations of the currents, $\nabla_\mu \theta^{\mu\nu} = D_\mu s^\mu = 0$, are equivalent to the local gauge transformation invariance of the external supergravity system,

$$\delta g_{\mu\nu}(x) = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \quad \delta \psi_\mu(x) = D_\mu \chi(x). \quad (7)$$

Furthermore, the covariant conservation of the chiral (or axial) vector current j_μ and the vanishing of both the γ -trace of supersymmetry current and the trace of energy-momentum tensor at classical level,

$$\nabla_\mu j^\mu = \gamma^\mu s_\mu = T^\mu_\mu = 0, \quad (8)$$

mean the Weyl transformation invariance of $g_{\mu\nu}$, the super-Weyl symmetry and the chiral gauge symmetry of the external conformal supergravity system,

$$\delta g_{\mu\nu} = g_{\mu\nu} \sigma(x), \quad \delta \psi_\mu = \gamma_\mu \eta(x), \quad \delta A_\mu(x) = \partial_\mu \Lambda(x). \quad (9)$$

This means that the classical superconformal symmetry of the supersymmetric gauge theory is equivalent to that of the external conformal supergravity. Therefore, in the context of the *AdS/CFT* (or more generally gravity/gauge) correspondence the superconformal anomaly in $\mathcal{N} = 1$ supersymmetric gauge theory due to the supergravity external sources will be reflected in the explicit violations of the bulk symmetries of $\mathcal{N} = 2$ gauged *AdS*₅ supergravity on the boundary [5, 23, 24, 25].

With no consideration on the quantum correction from the dynamics of the supersymmetric gauge theory, the external superconformal anomaly is exhausted at one-loop level. As given in Ref. [21], for a general $\mathcal{N} = 1$ supersymmetric gauge theory with N_v vector and N_χ chiral multiplets in an external supergravity background, the chiral R-symmetry and the Weyl anomalies read

$$\begin{aligned} \nabla_\mu j^\mu &= \frac{c-a}{24\pi^2} R_{\mu\nu\lambda\rho} \tilde{R}^{\mu\nu\lambda\rho} + \frac{5a-3c}{9\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}, \\ T^\mu_\mu &= \frac{c}{16\pi^2} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} - \frac{a}{16\pi^2} \tilde{R}_{\mu\nu\lambda\rho} \tilde{R}^{\mu\nu\lambda\rho} + \frac{c}{6\pi^2} F_{\mu\nu} F^{\mu\nu}, \\ \gamma_\mu s^\mu &= \left(AR_{\mu\nu\lambda\rho} \gamma^{\lambda\rho} + BF_{\mu\nu} \right) D^\mu \psi^\nu. \end{aligned} \quad (10)$$

The coefficients a and c are purely determined by the field contents. For a $\mathcal{N} = 1$ supersymmetric theory in the weak coupling limit, they are, respectively, [21]

$$c = \frac{1}{24} (3N_v + N_\chi), \quad a = \frac{1}{48} (9N_v + N_\chi). \quad (11)$$

The coefficients A and B in the super-Weyl anomaly $\gamma_\mu s^\mu$ are relevant to a and c . In above equations, $\gamma_{\mu\nu} = i/2 [\gamma_\mu, \gamma_\nu]$; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength corresponding to the external $U_R(1)$ vector field A_μ ; $R_{\mu\nu\lambda\rho}$ and $C_{\mu\nu\lambda\rho}$ are the Riemannian and Weyl tensors corresponding to the gravitational background field $g_{\mu\nu}$; $\tilde{R}_{\mu\nu\lambda\rho}$ and $\tilde{F}_{\mu\nu}$ are the Hodge duals of the Riemannian tensor and gauge field strength; D_μ is the covariant derivative with respect to both the external gravitational and gauge fields.

3 Five-dimensional gauged supergravity and *AdS*₅ boundary Reduction

In this section we explain why the superconformal anomaly of a four-dimensional supersymmetric gauge theory in a conformal supergravity background can be extracted from a gauged supergravity in five dimensions [6, 7, 8].

First, all the $\mathcal{N} = 2, 4, 8$ five-dimensional gauged supergravities admit an AdS_5 classical solution [6, 7, 8],

$$ds^2 = \frac{l^2}{r^2} \left(\eta_{\mu\nu} dx^\mu dx^\nu - dr^2 \right) \quad (12)$$

all the other fields vanishing. The cosmological term leading above solution comes from the value of the scalar potential at the critical point. Further, checking the supersymmetric transformation of the fermionic field in this background, one can find the non-vanishing Killing spinor [6, 7, 8]. Thus the AdS_5 solution preserve the full supersymmetry of the gauged supergravity.

Second, we choose this AdS_5 classical solution as the vacuum configuration of the five-dimensional gauged supergravity and investigate the corresponding dynamical features around such a vacuum background. For the $\mathcal{N} = 2$ $U(1)$ gauged supergravity, the Lagrangian density near the AdS_5 vacuum up to the quadratic terms in spinor fields is of the form [26]

$$\begin{aligned} 8\pi G^{(5)} E^{-1} \mathcal{L} = & -\frac{1}{2} \mathcal{R} - \frac{1}{2} \bar{\Psi}_M^i \Gamma^{MNP} D_N \Psi_{Pi} - \frac{3l^2}{32} \mathcal{F}_{MN} \mathcal{F}^{MN} - \frac{6}{l^2} \\ & - \frac{il^3}{64} E^{-1} \epsilon^{MNPQR} \mathcal{F}_{MN} \mathcal{F}_{PQ} \mathcal{A}_R - \frac{3i}{4l} \bar{\Psi}_M^i \Gamma^{MN} \Psi^{Nj} \delta_{ij} \\ & - \frac{3il}{32} \left(\bar{\Psi}_M^i \Gamma^{MNPQ} \Psi_{Ni} \mathcal{F}_{PQ} + 2 \bar{\Psi}^{Mi} \Psi_i^N \mathcal{F}_{MN} \right), \end{aligned} \quad (13)$$

where \mathcal{R} is the five-dimensional Riemannian scalar curvature; Ψ_M^i are the gravitini, $i = 1, 2$ are the $SU(2)$ R symmetry group indices; \mathcal{A}_M and \mathcal{F}_{MN} are the $U(1)$ gauge field and field strength; D_M is the covariant derivative with respect to the (modified) spin connection, the Christoffel and \mathcal{A}_M ; $G^{(5)}$ is the five-dimensional gravitational constant. The supersymmetry transformations at the leading order in spinor fields read [26]

$$\begin{aligned} \delta E_M^A &= \frac{1}{2} \bar{\mathcal{E}}^i \Gamma^A \Psi_{Mi}, \quad \delta \mathcal{A}_M = \frac{i}{l} \bar{\Psi}_M^i \mathcal{E}_i, \\ \delta \Psi_M^i &= D_M \mathcal{E}^i + \frac{il}{16} \left(\Gamma_M^{NP} - 4 \delta_M^N \Gamma^P \right) \mathcal{F}_{NP} \mathcal{E}^i + \frac{i}{2l} \Gamma_M \delta^{ij} \mathcal{E}_j. \end{aligned} \quad (14)$$

The investigation on the classical dynamics of the gauged supergravity around the AdS_5 vacuum configuration means looking for the solution to the classical equation of motion which should asymptotically approach the AdS_5 solution (12). Geometrically, this is actually a process of revealing the asymptotic dynamical behaviour of the bulk fields near the boundary of AdS_5 space-time. The procedure of doing this is straightforward [27]. As the first step, one should perform fix the local symmetries such as the local Lorentz symmetry, gauge symmetry and supersymmetry in the radial (fifth) direction in a way consistent with the AdS_5 classical solution. Then one should reveal the radial coordinate dependence near the boundary of AdS_5 space-time of the solutions to the classical equations of motion of the fields. There are some delicate points for the fermionic fields like the gravitino. The fermionic fields are the symplectic Majorana spinors in five dimensions, one should show how they reduce to chiral spinors in four dimensions. Specifically, most of the fields in the five-dimensional gauged supergravity belong to certain representation of R -symmetry, while the gauged supergravity in the bulk and the conformal supergravity on the boundary have different R -symmetries. Thus one should exhibit how the R -symmetry in the bulk converts into the one on the boundary. For the $\mathcal{N} = 2$ gauged supergravity,

the solutions to the classical equations of motion display the follow leading-order dependence on the radial coordinate near the AdS_5 boundary [26],

$$\begin{aligned}\mathcal{A}_\mu(x, r) &= A_\mu(x) + \mathcal{O}(r), \quad E_\mu^a(x, r) = \frac{l}{r} e_\mu^a(x) + \mathcal{O}(r), \quad E_r^{\bar{r}} = \frac{l}{r}, \\ \Psi_\mu^R &= \left(\frac{2l}{r}\right)^{1/2} \psi_\mu^R(x), \quad \Psi_\mu^L = (2lr)^{1/2} \chi_\mu^L(x), \\ \chi_\mu^L &= \frac{1}{3} \gamma^\nu \left(D_\mu \psi_\nu^R - D_\nu \psi_\mu^R \right) - \frac{i}{12} \epsilon_{\mu\nu}{}^{\lambda\rho} \gamma_5 \gamma^\nu \left(D_\lambda \psi_\rho^R - D_\rho \psi_\lambda^R \right),\end{aligned}\quad (15)$$

and all other fields vanish. In above equations, $\mu, a = 0, \dots, 3$ are the Riemannian and local Lorentz indices on the boundary, respectively, and \bar{r} is the Lorentz index in the radial direction. The various quantities including the γ -matrices and the covariant derivative reduced from the five-dimensional case are the following [26],

$$\begin{aligned}\gamma_a &= \Gamma_a, \quad \Gamma_\mu = E_\mu^a \Gamma_a = \frac{l}{r} \gamma_\mu, \quad \Gamma^\mu = \bar{E}^\mu_a \Gamma^a = \frac{r}{l} \gamma^\mu, \\ \gamma_5 &= i\Gamma^{\bar{r}} = -i\Gamma_{\bar{r}}, \quad \gamma_5^2 = 1; \quad D_\mu(x) \equiv \nabla_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} - \frac{3}{4} A_\mu \gamma_5, \\ \Psi_\mu &\equiv \Psi_{\mu 1} + i\Psi_{\mu 2}, \quad \Psi_\mu^R \equiv \frac{1}{2}(1 - \gamma_5)\Psi_\mu, \quad \Psi_\mu^L \equiv \frac{1}{2}(1 + \gamma_5).\end{aligned}\quad (16)$$

Redefining the bulk supersymmetry transformation parameter, $\mathcal{E}(x, r) = \mathcal{E}_1(x, r) + i\mathcal{E}_2(x, r)$, decomposing it into the chiral components, and further choosing radial coordinate dependence of $\mathcal{E}^{L,R}$ in the same way as the bulk gravitino,

$$\mathcal{E}^R(x, r) = \left(\frac{2l}{r}\right)^{1/2} \epsilon^R(x), \quad \mathcal{E}^L(x, r) = (2lr)^{1/2} \eta^L(x), \quad (17)$$

one can find that the bulk supersymmetry transformation reduces to that for $\mathcal{N} = 1$ conformal supergravity in four dimensions with ϵ and η playing the roles of parameters for supersymmetry and special supersymmetry transformations, respectively [22, 26],

$$\delta e_\mu^a = -\frac{1}{2} \bar{\psi}_\mu \gamma^a \epsilon, \quad \delta \psi_\mu = \nabla_\mu \epsilon - \frac{3}{4} A_\mu \gamma_5 \epsilon - \gamma_\mu \eta, \quad \delta A_\mu = i \left(\bar{\psi}_\mu \gamma_5 \eta - \bar{\chi}_\mu \gamma_5 \epsilon \right), \quad (18)$$

where all the spinorial quantities, $\psi_\mu(x)$, $\chi_\mu(x)$, $\epsilon(x)$ and $\eta(x)$ are Majorana spinors constructed from their chiral components $\psi_\mu^R(x)$, $\chi_\mu^L(x)$, $\epsilon^R(x)$ and $\eta^L(x)$.

As for other local symmetries of five-dimensional gauged supergravity, it has been proved that for any domain wall solution of the following form which asymptotically approaches the AdS_5 solution (12),

$$ds^2 = G_{MN} dX^M dX^N = \frac{l^2}{r^2} \left[g_{\mu\nu}(x, r) dx^\mu dx^\nu - dr^2 \right], \quad (19)$$

the diffeomorphism symmetry preserving its above form must be a combination of the four-dimensional diffeomorphism symmetry and the Weyl symmetry [28],

$$\delta g_{\mu\nu}(x, r) = 2\sigma(x) \left(1 - \frac{1}{2} r \partial_r \right) g_{\mu\nu}(x, r) + \nabla_\mu \xi_\nu(x, r) + \nabla_\nu \xi_\mu(x, r). \quad (20)$$

The $U(1)$ bulk gauge symmetry under the transformation $\delta\mathcal{A}_M(x, r) = \partial_M\Lambda(x, r)$, automatically reduces to the $U(1)$ chiral (or equivalently axial) vector gauge symmetry on the AdS_5 boundary.

The above fact indicates that the on-shell five-dimensional gauged supergravity near the AdS_5 vacuum configuration leads to the off-shell conformal supergravity on the AdS_5 boundary. Therefore, this has provided the justification that the superconformal anomaly of a supersymmetric gauge theory in a conformal supergravity background can be extracted from the five-dimensional gauged supergravity.

4 Holographic Superconformal Anomaly

4.1 Holographic Chiral Anomaly

The holographic origin of the R-symmetry anomaly is the Chern-Simons (CS) five-form term in the gauge supergravity [5, 29]. For the $\mathcal{N} = 8$ $SO(6) \cong SU(4)$ gauged supergravity in five dimensions, the CS term is [6]

$$\begin{aligned} S_{\text{CS}}[\mathcal{A}] &= \frac{l^3}{48\pi G^{(5)}} \int \text{Tr} \left[\mathcal{A}(d\mathcal{A})^2 + \frac{3}{2}\mathcal{A}^3 d\mathcal{A} + \frac{3}{5}\mathcal{A}^5 \right] \\ &= \frac{l^3}{48\pi G^{(5)}} \int \text{Tr} \left(\mathcal{A}\mathcal{F}^2 - \frac{1}{2}\mathcal{A}^3\mathcal{F} + \frac{1}{10}\mathcal{A}^5 \right) \\ &= \frac{l^3}{192\pi G^{(5)}} \int d^5x \epsilon^{MNPQR} d^{abc} \left(\mathcal{A}_M^a \mathcal{F}_{NP}^b \mathcal{F}_{QR}^c - f^{ade} \mathcal{A}_M^d \mathcal{A}_N^e \mathcal{A}_P^b \mathcal{F}_{QR}^c \right. \\ &\quad \left. + \frac{2}{5} f^{ade} f^{efg} \mathcal{A}_M^d \mathcal{A}_N^f \mathcal{A}_P^g \mathcal{A}_Q^b \mathcal{A}_R^c \right), \end{aligned} \quad (21)$$

where \mathcal{A}_M and \mathcal{F}_{MN} are the $SU(4)$ gauge field and the field strength. A CS term has a particular feature: its gauge variation is a total derivative. Therefore, under the bulk gauge transformation,

$$\delta\mathcal{A}_M^a(x, r) = [D_M V(x, r)]^a, \quad (22)$$

$V(x, r) = V^a(x, r)t^a$ being a gauge transformation parameter, the other gauge field relevant terms are gauge invariant, but the gauge transformation of CS term leaves a total derivative term,

$$\begin{aligned} \delta_V S_{\text{SUGRA}}[\mathcal{A}, \dots] &= \delta_V S_{\text{CS}}[\mathcal{A}] = \int dQ_4^1(V, \mathcal{A}) \\ &= \frac{l^3}{48\pi G^{(5)}} \int d\text{Tr} \left[V d \left(A dA + \frac{1}{2} A^3 \right) \right]. \end{aligned} \quad (23)$$

Choosing the boundary behaviour of the bulk gauge transformation parameter as the bulk gauge field, $V(x, r)|_{r \rightarrow 0} = v(x)$ and making use of the AdS/CFT correspondence (3),

$$\begin{aligned} \delta_V S_{\text{SUGRA}}[\mathcal{A}, \dots]|_{\mathcal{A}_M \rightarrow A_\mu, V \rightarrow v} &= \delta_v \Gamma_{\text{SYM}}[A_\mu^a, \dots] = \int d^4x \frac{\delta\Gamma}{\delta A_\mu^a(x)} \delta A_\mu^a(x) \\ &= \int d^4x j^{a\mu} \delta A_\mu^a(x) = \int d^4x j^{a\mu}(x) [D_\mu v(x)]^a = - \int d^4x v^a(x) [D_\mu j^\mu(x)]^a. \end{aligned} \quad (24)$$

one can obtain from Eqs. (23) and (24)

$$[D^\star j(x)]^a = -\frac{l^3}{48\pi G^{(5)}} \text{Tr} t^a \left[F^2 - \frac{1}{2} (A^2 F + F A^2 + A F A) + \frac{1}{2} A^4 \right]. \quad (25)$$

Considering the following relations among the AdS_5 radius l , string coupling g_s , the number N of $D3$ -branes, the five- and ten-dimensional gravitational constants related by the compactification of the type IIB supergravity on S^5 of radius l [10],

$$G^{(5)} = \frac{G^{(10)}}{\text{Volume}(S^5)} = \frac{G^{(10)}}{l^5 \pi^3}, \quad G^{(10)} = 8\pi^6 g_s^2, \quad l = (4\pi N g_s)^{1/4}, \quad (26)$$

one immediately recognizes the holographic Bardeen (consistent) anomaly,

$$[D_\mu j^\mu(x)]^a = -\frac{N^2}{24\pi^2} e^{-1} \epsilon^{\mu\nu\lambda\rho} \partial_\mu \text{Tr} t^a \left(A_\nu \partial_\lambda A_\rho + \frac{1}{2} A_\nu A_\lambda A_\rho \right). \quad (27)$$

For the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, its R -symmetry group is $U(2)_R \cong SU(2)_R \times U(1)_R$. It is the $U(1)_R$ that becomes anomalous. The dual gravitational theory is the five-dimensional $SU(2) \times U(1)$ gauged $\mathcal{N} = 4$ supergravity. The holographic chiral $U(1)_R$ anomaly comes from the $SU(2) \times U(1)$ -mixed CS term in the gauged supergravity [7, 9],

$$S_{\text{CS}}[\mathcal{W}, \mathcal{A}, \dots] = \frac{l^3}{64\pi G^{(5)}} \int \text{Tr} (\mathcal{G} \wedge \mathcal{G}) \wedge \mathcal{A}, \quad (28)$$

where \mathcal{W} and \mathcal{A} are the $SU(2)$ and $U(1)$ gauge fields, and \mathcal{G} the $SU(2)$ field strength. The reduction of the bulk $U(1)$ gauge transformation $\delta\mathcal{A} = dV$ to the AdS_5 boundary leads to the holographic $U_R(1)$ anomaly,

$$\partial_\mu (e j^\mu) = -\frac{N^2}{32\pi^2} \epsilon^{\mu\nu\lambda\rho} \text{Tr} (G_{\mu\nu} G_{\lambda\rho}). \quad (29)$$

The justification that the boundary value $A_\mu(x)$ of the bulk gauge field \mathcal{A}_M is considered as the external chiral (or axial) gauge field in four dimensions is implied from the boundary reductions of the bulk covariant derivative and of the supersymmetric transformation listed in Eqs. (16) and (18).

However, Eqs. (27) and (29) do not contain the gravitational background contribution shown in the general expression (10). In fact, for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, there exists no gravitational contribution. The reason is that the field contents of $\mathcal{N} = 4$ SYM can be considered as one $\mathcal{N} = 1$ vector multiplet plus three chiral multiplets in the adjoint representation of $SU(N)$ and according to Eq. (11) this yields $c = a = (N^2 - 1)/4$ [30]. Thus the CS term composed of the $SU_R(4)$ gauge field is fully responsible for the holographic source of the chiral R -symmetry anomaly. For the general $\mathcal{N} = 1, 2$ supersymmetric gauge theories, Eq. (11) shows that usually $a \neq c$, hence the gravitational background contribution to the chiral anomaly should arise. Its absence implies that the five-dimensional gauged supergravity (or the type IIB supergravity in $AdS_5 \times X^5$ background) is only the lowest order approximation to type IIB superstring theory) and corresponds only to the leading order of the large- N expansion of supersymmetric gauge theory. In Ref. [31] it was shown that for an $\mathcal{N} = 2$ supersymmetric $USp(2N)$ gauge theory coupled to two hypermultiplets in the fundamental and antisymmetric tensor representations of the gauge group, respectively, the gravitational background part of the holographic chiral anomaly does originate from a mixed CS term. However, this CS term is obtained from the compactification on S^3 of the Wess-Zumino term describing the interaction of the R-R 4-form field with eight $D7$ -branes and one orientifold 7-plane system. Specifically, this gravitational background term is at the subleading N -order rather than the leading N^2 -order in

the large- N expansion of the $\mathcal{N} = 2$ supersymmetric $USp(2N)$ gauge theory. This fact exposes the limitation of the gauged supergravity in providing an equivalent physical description to the supersymmetric gauge theory.

4.2 Holographic Weyl Anomaly

The holographic origin of the Weyl anomaly of a supersymmetric gauge theory lies in the AdS_5 boundary behaviour of the on-shell action of the gauged supergravity. Due to the infinity of the boundary, the on-shell action of the five-dimensional gauged supergravity in AdS_5 background suffers from the infrared divergences when approaching the boundary. Therefore, one must perform a so-called “holomorphic renormalization” [24]. That is, first introducing an IR cut-off when one integrate over the radial (fifth) coordinate to evaluate the on-shell action, then similar to dealing with the UV divergence in a renormalizable quantum field theory, defining a counterterm according to a preferred renormalization condition to cancel the IR divergence, finally removing the cut-off to get the renormalized on-shell action for the gauged supergravity. Specifically, Eq. (20) shows that the bulk diffeomorphism symmetry of the gauged supergravity decomposes into the diffeomorphism symmetry and the Weyl symmetry on the boundary [28]. These two symmetries cannot be preserved simultaneously in implementing the holomorphic renormalization. Thus if one requires the diffeomorphism symmetry preserved, the holographic Weyl anomaly of a supersymmetric gauge theory will arise.

We take the $\mathcal{N} = 8$ $SO(6)$ gauged supergravity in five dimensions as an example, and choose the truncated action consisting only of the Einstein-Hilbert action and the non-vanishing scalar potential [23, 24],

$$8\pi G^{(5)} E^{-1} \mathcal{L}_{\text{trunc}} = -\frac{1}{2} \mathcal{R} - P[\phi]. \quad (30)$$

The corresponding Einstein equation is

$$\mathcal{R}_{MN} - \frac{1}{2} \mathcal{R} G_{MN} = P[\phi = 0] G_{MN}. \quad (31)$$

The solution to the Einstein equation (31) is the domain wall (19). It should be emphasized that the existence of scalar field is necessary for the domain wall solution (19). Otherwise there will be no non-trivial vacuum configurations and the domain wall solution does not exist. Near the AdS_5 boundary ($r \rightarrow 0$), the solution $g_{\mu\nu}(x, r)$ admits the following expansion [23, 24],

$$\begin{aligned} g_{\mu\nu}(x, r) &= g_{(0)\mu\nu}(x) + g_{(2)\mu\nu}(x) \frac{r^2}{l^2} \\ &+ \left[g_{(4)\mu\nu} + h_{1(4)\mu\nu} \ln \frac{r^2}{l^2} + h_{2(4)\mu\nu} \left(\ln \frac{r^2}{l^2} \right)^2 \right] \left(\frac{r^2}{l^2} \right)^2 + \dots \end{aligned} \quad (32)$$

Substituting (32) into the Einstein equation (31) with the cosmological constant provided by the value of the scalar potential at the critical point $\phi = 0$, one can determine the coefficients $g_{(2k)\mu\nu}$, $h_{\mu\nu}$ in terms of $g_{(0)\mu\nu}$ [23, 24],

$$g_{(2)\mu\nu} = \frac{l^2}{2} \left(R_{\mu\nu} - \frac{1}{6} R g_{(0)\mu\nu} \right),$$

$$\begin{aligned}
h_{1(4)\mu\nu} &= \frac{l^4}{8} \left(R_{\mu\lambda\nu\rho} R^{\lambda\rho} + \frac{1}{6} \nabla_\mu \nabla_\nu R - \frac{1}{2} \nabla^2 R_{\mu\nu} - \frac{1}{3} R R_{\mu\nu} \right) \\
&\quad + \frac{l^4}{32} g_{(0)\mu\nu} \left(\frac{1}{3} \nabla^2 R + \frac{1}{3} R^2 - R_{\lambda\rho} R^{\lambda\rho} \right), \\
h_{2(4)\mu\nu} &= 0, \quad \text{Tr} g_{(4)} = \frac{1}{4} \text{Tr} \left(g_{(2)} \right)^2, \\
\nabla^\nu g_{(4)\mu\nu} &= \nabla^\nu \left\{ -\frac{1}{8} \left[\text{Tr} g_{(2)}^2 - \left(\text{Tr} g_{(2)} \right)^2 \right] g_{(0)\mu\nu} + \frac{1}{2} \left(g_{(2)}^2 \right)_{\mu\nu} - \frac{1}{4} g_{(2)\mu\nu} \text{Tr} g_{(2)} \right\}, \quad (33)
\end{aligned}$$

where $\text{Tr} g_{(2)} = g_{(0)}^{\mu\nu} g_{(2)\mu\nu}$, $R_{\mu\nu\lambda\rho}$, $R_{\mu\nu}$ and R are the Riemannian tensor, Ricci tensor and curvature scalar corresponding to $g_{(0)\mu\nu}$, respectively. Inserting above solution into Eq. (30) to evaluate the on-shell action, one finds that it is divergent when approaching the AdS_5 boundary. Thus one must perform the radial integration in a finite domain by introducing a cut-off $r = \epsilon > 0$ and get the regularized on-shell action,

$$\begin{aligned}
S_{\text{reg}} &= \frac{1}{8\pi G^{(5)}} \int_{r=\epsilon>0} d^5 X E(x, r) \left(-\frac{1}{2} \mathcal{R} - P[\phi = 0] \right) \\
&= \frac{l^5}{8\pi G^{(5)}} \int_\epsilon \frac{dr}{r^5} \int d^4 x \sqrt{g(x, r)} \frac{2}{3} P[\phi = 0] \\
&= -\frac{l^5}{8\pi G^{(5)}} \int_\epsilon \frac{dr}{r^5} \int d^4 x \sqrt{g(x, r)} \frac{4}{3} \Lambda = \frac{l^3}{\pi G^{(5)}} \int_\epsilon \frac{dr}{r^5} \int d^4 x \sqrt{g(x, r)} \\
&= \frac{l^3}{\pi G^{(5)}} \int d^4 x \sqrt{g_0(x)} \int_\epsilon \frac{dr}{r^5} \left\{ 1 + \frac{r^2}{2l^2} \text{Tr} g_{(2)} + \frac{r^4}{4l^4} \left[\left(\text{Tr} g_{(2)} \right)^2 - \text{Tr} g_{(2)}^2 \right] + \dots \right\} \\
&= \frac{l^3}{\pi G^{(5)}} \int d^4 x \sqrt{g_{(0)}} \left[-\frac{1}{4\epsilon^4} - \frac{1}{12\epsilon^2} R - \frac{1}{16} \ln \frac{\epsilon}{l} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) + \mathcal{L}_{\text{finite}} \right] \\
&= S_{\epsilon^{-4}} + S_{\epsilon^{-2}} + S_{\ln \epsilon} + S_{\text{finite}}. \quad (34)
\end{aligned}$$

In writing down (34) the Einstein equation (31), the identification $\Lambda = -2P[\phi = 0] = -6/l^2$ and the matrix operation

$$\begin{aligned}
\sqrt{\det(1 + A)} &= \exp \left[\frac{1}{2} \text{Tr} \ln(1 + A) \right] = \exp \left[\frac{1}{2} \text{Tr} \left(A - \frac{1}{2} A^2 + \dots \right) \right] \\
&= 1 + \frac{1}{2} \text{Tr} A + \frac{1}{4} \left[(\text{Tr} A)^2 - \text{Tr} A^2 \right] + \dots, \quad (35)
\end{aligned}$$

have been employed. \mathcal{L}_{fin} consists of the terms standing the $\epsilon \rightarrow 0$ limit.

To get the renormalized on-shell action, one must first define a subtracted action by introducing the counterterms to cancel the IR divergence in the limit $\epsilon \rightarrow 0$,

$$S_{\text{sub}}[g_{(0)\mu\nu}, \epsilon^2/l^2, \dots] = S_{\text{reg}} + S_{\text{counter}} \quad (36)$$

A holographically renormalized on-shell action is yielded after removing the regulator,

$$S_{\text{ren}}[g_{(0)\mu\nu}, \dots] = \lim_{\epsilon \rightarrow 0} S_{\text{sub}}[g_{(0)\mu\nu}, \epsilon^2/l^2, \dots] \quad (37)$$

In adding the counterterm, there arise a finite ambiguity similar to cancelling the UV divergence in a perturbative quantum field theory. This ambiguity can be fixed by the symmetry

requirement. According to Eq. (20), the bulk diffeomorphism transformation converts into a four-dimensional Weyl and a diffeomorphism transformations near the AdS_5 boundary. Requiring the four-dimensional diffeomorphism symmetry preserved in performing subtraction, one can introduce the following counterterm similar to the minimal subtraction in the dimensional regularization of the perturbative quantum field theory [23, 24],

$$S_{\text{counter}} = \frac{l^3}{\pi G^{(5)}} \int d^4x \sqrt{g_{(0)}} \left[\frac{1}{4\epsilon^4} + \frac{1}{12\epsilon^2} R + \frac{1}{16} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) \ln \frac{\epsilon}{l} \right], \quad (38)$$

and consequently, the renormalized on-shell action is just the finite part of the regularized one,

$$S_{\text{ren}} = \frac{l^3}{\pi G^{(5)}} \int d^4x \sqrt{g_{(0)}} \mathcal{L}_{\text{finite}}. \quad (39)$$

There are several ways to extract the Weyl anomaly from the renormalized action [23, 24]. The most straightforward way is considering the scale transformation of the regularized action, i.e., choosing the parameter $\sigma(x)$ of the Weyl transformation as a constant σ . The regularized action is invariant under the combination of two transformations, $\delta g_{(0)\mu\nu} = 2\sigma g_{(0)\mu\nu}$, $\delta\epsilon = 2\sigma\epsilon$. That is [23],

$$(\delta_{g_0} + \delta_\epsilon) S_{\text{reg}} = (\delta_{g_0} + \delta_\epsilon) (S_{\epsilon^{-4}} + S_{\epsilon^{-2}} + S_{\ln \epsilon} + S_{\text{finite}}) = 0. \quad (40)$$

However, it has been found that [23]

$$\delta_{g_0} (S_{\epsilon^{-4}} + S_{\epsilon^{-2}}) = 0, \quad \delta_\epsilon (S_{\epsilon^{-4}} + S_{\epsilon^{-2}}) = 0, \quad \delta_{g_0} S_{\ln \epsilon} = 0, \quad \delta_\epsilon S_{\text{finite}} = 0. \quad (41)$$

This leads to

$$\begin{aligned} \delta_{g_0} S_{\text{finite}} &= \delta_{g_0} S_{\text{ren}} = \int d^4x \sqrt{g_{(0)}} \langle T^\mu_\mu \rangle \sigma = -\delta_\epsilon S_{\ln \epsilon} \\ &= \frac{l^3}{8\pi G^{(5)}} \int d^4x \sqrt{g_{(0)}} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) \sigma, \end{aligned} \quad (42)$$

and yields the Weyl anomaly in the gravitational background [23, 24],

$$\langle T^\mu_\mu \rangle = \frac{N^2}{4\pi^2} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right). \quad (43)$$

It can be further rewritten as the combination of the A - and B -type anomalies, i.e, the sum of the Euler number density and the square of the Weyl tensor [23, 32],

$$\begin{aligned} \langle T^\mu_\mu \rangle &= -\frac{N^2}{\pi^2} (E_4 + W_4), \\ E_4 &= \frac{1}{8} \tilde{R}_{\mu\nu\lambda\rho} \tilde{R}^{\mu\nu\lambda\rho} = \frac{1}{8} (R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} - 4R^{\mu\nu} R_{\nu\mu} + R^2), \\ W_4 &= -\frac{1}{8} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} = -\frac{1}{8} (R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} - 2R^{\mu\nu} R_{\mu\nu} + \frac{1}{3} R^2). \end{aligned} \quad (44)$$

It is the trace anomaly of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in the external gravitational field at the leading order of large- N expansion [23].

The gauge field contribution to the Weyl anomaly can be similarly calculated when switching on the gauge field sector of the $\mathcal{N} = 8$ gauged supergravity in five dimensions [24]. If one considers only the bilinear terms in the gauge fields, the $SO(6)$ gauge field can be approximated by some uncoupled Abelian sectors \mathcal{A}_M [33]. The solution to the gauge field equation near the AdS_5 boundary is [24]

$$\mathcal{A}_\mu(x, r) = A_\mu(x) + \frac{r^2}{l^2} \left[A_{(2)\mu}(x) + \tilde{A}_{(2)\mu}(x) \ln \frac{r^2}{l^2} \right] + \dots, \quad (45)$$

where $A_{(2)\mu}(x)$ and $\tilde{A}_{(2)\mu}(x)$ can be expressed as the functional of $A_\mu(x)$ when inserting Eq. (45) into the classical equation of motion for \mathcal{A}_μ . The regularized action of the gauge field sector is [24]

$$S_{\text{reg}} = \frac{l^3}{8\pi G^{(5)}} \int d^4x \sqrt{g_{(0)}} \left[\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \ln \frac{\epsilon}{l} + \mathcal{L}_{\text{finite}} \right] \quad (46)$$

The gauge field part of the Weyl anomaly given in Eq. (10) can be derived in the same way as the gravitational case.

4.3 Holographic Supersymmetry Current Anomaly

The production of the super-Weyl anomaly of a four-dimensional supersymmetric gauge theory from the five-dimensional gauged supergravity lies in two aspects. First, as a supersymmetric theory, the supersymmetry transformation of the Lagrangian of the gauged supergravity must be composed of the total derivative terms. These terms cannot be naively ignored due to the existence of the boundary AdS_5 . Second, near the AdS_5 boundary the bulk supersymmetry transformation decomposes into the four-dimensional supersymmetry and super-Weyl transformations as shown in Eq. (18). If we require four-dimensional supersymmetry preserved on the boundary, the total derivative terms should yield the anomaly of the supersymmetry current via the AdS/CFT correspondence. Therefore, the key point is to calculate the supersymmetric variation of the gauged supergravity and get the total derivative terms. Then putting these terms on the AdS_5 boundary, one should find the holographic supersymmetry current anomaly.

We have worked out these total derivative terms of the simplest case in the five dimensional gauged supergravities, the $\mathcal{N} = 2$ $U(1)$ gauged supergravity whose Lagrangian is given in Eq. (13). The concrete calculation is very lengthy and has been displayed in a great detail in Ref. [25]. The variation of the Lagrangian (13) under the supersymmetric transformation (14) yields

$$\begin{aligned} \delta S = & \frac{1}{8\pi G^{(5)}} \int d^5x E \nabla_M \left(-\frac{9il}{16} \bar{\mathcal{E}}^i \Psi_{Ni} \mathcal{F}^{MN} - \frac{1}{2} \bar{\mathcal{E}}^i \Gamma^{MNP} \nabla_N \Psi_{Pi} \right. \\ & + \frac{3}{8} \bar{\mathcal{E}}^i \Gamma^{MNP} \Psi_P^j \delta_{ij} \mathcal{A}_N - \frac{3il}{32} E^{-1} \epsilon^{MNPQR} \bar{\mathcal{E}}^i \Gamma_R \Psi_{Ni} \mathcal{F}_{PQ} \\ & \left. + \frac{9}{4} \bar{\mathcal{E}}^i \Gamma^{MN} \Psi_N^i \delta_{ij} + \frac{l^2}{16} E^{-1} \epsilon^{MNPQR} \bar{\mathcal{E}}^i \Psi_R \mathcal{A}_N \mathcal{F}_{PQ} \right). \end{aligned} \quad (47)$$

In deriving above derivative terms, one must bear in mind that in the Lagrangian (13) the $U(1)$ gauge field is imaginary and the gravitino field is an $SU(2)$ symplectic Majorana spinor,

$$\mathcal{A}_M^* = -\mathcal{A}_M, \quad \Psi^i = C^{-1} \Omega^{ij} \bar{\Psi}_j^T = C^{-1} \bar{\Psi}^{iT}, \quad \bar{\Psi}^i = -\Psi^{iT} C,$$

$$\begin{aligned}
& \bar{\Psi}^i \Gamma_{M_1 \dots M_n} \Phi_i = -\Psi^{iT} C \Gamma_{M_1 \dots M_n} C^{-1} \bar{\Phi}_i^T \\
= & \begin{cases} -\Psi^{iT} \Gamma_{M_1 \dots M_n} \bar{\Phi}_i^T = \bar{\Phi}_i \Gamma_{M_1 \dots M_n} \Psi^i = -\bar{\Phi}^i \Gamma_{M_1 \dots M_n} \Psi_i, & n = 0, 1, 4, 5, \\ \Psi^{iT} \Gamma_{M_1 \dots M_n} \bar{\Phi}_i^T = -\bar{\Phi}_i \Gamma_{M_1 \dots M_n} \Psi^i = \bar{\Phi}^i \Gamma_{M_1 \dots M_n} \Psi_i, & n = 2, 3, \end{cases} \quad (48)
\end{aligned}$$

The convention for the Γ -matrix in five dimensions and the frequently used relations are chosen as the following,

$$\begin{aligned}
\Gamma_{MN} &= \frac{1}{2} [\Gamma_M, \Gamma_N], \quad \Gamma^{MNP} = -\frac{1}{2!} E^{-1} \epsilon^{MNPQR} \Gamma_{QR}, \\
\Gamma^{MNPQ} &= E^{-1} \epsilon^{MNPQR} \Gamma_R, \quad \Gamma_{MNPQR} = E \epsilon_{MNPQR}. \\
\Gamma_{MN} \Gamma_{PQ} &= E \epsilon_{MNPQR} \Gamma^R - (G_{MP} G_{NQ} - G_{MQ} G_{NP}), \\
\Gamma_M \Gamma_{NP} &= \Gamma_{MNP} + G_{MN} \Gamma_P - G_{MP} \Gamma_N, \\
\Gamma^{MNP} \nabla_N \nabla_P \Psi_i &= \frac{1}{2} \Gamma^{MNP} [\nabla_N, \nabla_P] \Psi_i = \frac{1}{8} \Gamma^{MNP} \mathcal{R}_{NPAB} \Gamma^{AB} \Psi_i. \quad (49)
\end{aligned}$$

The following Ricci and Bianchi identities for the Riemannian curvature tensor and the $U(1)$ field strength are employed in the calculation,

$$\epsilon^{MNPQR} \mathcal{R}_{SPQR} = 0, \quad \epsilon^{MNPQR} \nabla_N \mathcal{R}_{STPQ} = 0, \quad \epsilon^{MNPQR} \nabla_N \mathcal{F}_{QR} = 0. \quad (50)$$

Specifically, due to the noncommutativity between ∇_M and $\Gamma_{M_1 \dots M_n}$, we reiteratively make the operation,

$$\Gamma_{M_1 \dots M_n} \nabla_M (\dots) = [\Gamma_{M_1 \dots M_n}, \nabla_M] (\dots) + \nabla_M [\Gamma_{M_1 \dots M_n} (\dots)]. \quad (51)$$

In this case it is convenient to choose the inertial coordinate system, i.e. the Christoffel symbol $\Gamma_{NP}^M = 0$. Consequently, the metricity condition leads to $\partial_M E_N^A = 0$, and hence the modified spin connection Ω_{MAB} contains only the quadratic fermionic terms. This simplifies the calculation greatly since we retain only the quadratic fermionic terms in calculating the supersymmetry variation.

The holographic super-Weyl anomaly can be extracted from above total derivative terms. First, we take into account the radial coordinate dependence of bulk fields and of the supersymmetry transformation parameter \mathcal{E}^i near the AdS_5 boundary as well as the connection between five- and four-dimensional γ -matrices listed in (15), (16) and (17). Second, to avoid the possible IR divergence due to the infinite AdS_5 boundary, we must integrate over the radial coordinate to the cut-off $r = \epsilon$ and then take the limit $\epsilon \rightarrow 0$. Finally, we use the fact that the metric on the boundary should be the induced metric [23, 24]

$$g_{\mu\nu}(x) = \left. \frac{l^2}{\epsilon^2} g_{\mu\nu}(x, \epsilon) \right|_{\epsilon \rightarrow 0} \quad (52)$$

rather than $g_{\mu\nu}(x, \epsilon)$ [24]. With all these considerations together, we have found that the non-vanishing contribution comes only from the term $E^{-1} \epsilon^{MNPQR} \bar{\mathcal{E}}^i \Gamma_R \Psi_{Ni} \mathcal{F}_{PQ}$. Therefore, we obtain [25]

$$\delta S = \frac{3il^3}{8 \times 32\pi G^{(5)}} \int d^4 x \epsilon^{\mu\nu\lambda\rho} F_{\nu\lambda} \bar{\eta} \gamma_\rho \chi_\mu, \quad (53)$$

where χ_μ is the Majorana spinor constructed from the left-handed spinor χ_μ^L given in (15).

Eq. (53) definitely leads to the super-Weyl anomaly in the context of AdS/CFT correspondence (3) since it is proportional to the special supersymmetry transformation parameter η . Inserting the explicit form χ_μ expressed in terms of the gravitino ψ_μ in four-dimensional $\mathcal{N} = 1$ conformal supergravity, we have

$$\begin{aligned}
\delta S &= \int d^4x \bar{\eta} \gamma^\mu s_\mu \\
&= \frac{3il^3}{8 \times 32\pi G^{(5)}} \int d^4x \epsilon^{\mu\nu\lambda\rho} F_{\nu\lambda} \bar{\eta} \gamma_5 \gamma_\rho \gamma^\alpha \left[\frac{1}{3} (D_\mu \psi_\alpha - D_\alpha \psi_\mu) - \frac{i}{6} \epsilon_{\mu\alpha\sigma\delta} \gamma_5 D^\sigma \psi_\delta \right] \\
&= -\frac{l^3}{8 \times 16\pi G^{(5)}} \int d^4x \left[F^{\mu\nu} D_\mu \psi_\nu + \epsilon^{\mu\nu\lambda\rho} \gamma_5 F_{\mu\nu} D_\lambda \psi_\rho \right. \\
&\quad \left. + \frac{1}{2} \sigma^{\mu\nu} F_{\nu\lambda} (D_\mu \psi^\lambda - D^\lambda \psi_\mu) \right], \tag{54}
\end{aligned}$$

where we have used the γ -matrix algebraic relations, $\gamma^\mu \gamma^\nu = g^{\mu\nu} - i\gamma^{\mu\nu}$, $\gamma_5 \gamma^{\mu\nu} = i\epsilon^{\mu\nu\lambda\rho} \gamma_{\lambda\rho}/2$. The gauge field part of the holographic super-Weyl anomaly of the $SU(N)$ supersymmetric gauge theory at the leading-order of the large- N expansion is thus yielded,

$$\gamma_\mu s^\mu = \frac{N^2}{64\pi^2} \left[F^{\mu\nu} D_\mu \psi_\nu + \epsilon^{\mu\nu\lambda\rho} \gamma_5 F_{\mu\nu} D_\lambda \psi_\rho + \frac{1}{2} \sigma^{\mu\nu} F_{\nu\lambda} (D_\mu \psi^\lambda - D^\lambda \psi_\mu) \right]. \tag{55}$$

There should also has a contribution from the external gravitational background shown in Eq. (10), which was found long time ago [34]. The reason for having failed to reproduce the gravitational part is not clear to us yet, we have the following two speculations based on the process of deriving the gravitational background parts in both the holographic Weyl and chiral anomalies [30, 31].

The first intuitive argument, as mentioned before, is that the five-dimensional gauged supergravity (or the type IIB supergravity in $AdS_5 \times X^5$ background) is only the lowest approximation to the type IIB superstring theory in $AdS_5 \times X^5$ background. Thus, it is possible that the gravitational part cannot be revealed within the five-dimensional gauged supergravity itself, and one must consider the higher-order gravitational action such as the Gauss-Bonnet term generated from the superstring theory [35]. The supersymmetry variation of the gauged supergravity containing the high-order gravitational term and the corresponding fermionic terms required by supersymmetry may lead to the gravitational contribution to the super-Weyl anomaly.

The other possible reason for the failure of getting the gravitational background contribution is that in Eq. (15) only the leading-order of radial coordinate dependence of the bulk fields near the AdS_5 boundary is taken into account. As shown above in deriving the holographic Weyl anomaly, when one makes a complete near-boundary analysis and considers the asymptotic expansion of the bulk fields in terms of the radial coordinate beyond the leading-order until the emergence of the logarithmic term [23, 24], the higher-order gravitational terms can appear in the on-shell action [23], and they lead to the holographic Weyl anomaly composed of the $R_{\mu\nu} R^{\mu\nu}$ and R^2 terms. Therefore, it is also possible that the gravitational background part in the super-Weyl anomaly can arise if one takes into account the logarithmic term in the expansion of the on-shell bulk fields. In this case the on-shell action of the five-dimensional gauged supergravity should have the infrared divergence when approaching the AdS_5 boundary. One must perform the holographic renormalization to get the renormalized on-shell action [24].

We have not realized whether there are any physical reasons for the difference between these two holographic contributions to the super-Weyl anomaly. The essence of the holographic anomaly is the anomaly inflow from the bulk theory to the AdS_5 boundary [36]. Thus the absence of the gravitational part might be relevant to the difference between the anomaly inflows contributed from the gravitational and gauge background fields.

5 Summary

We have reviewed how the superconformal anomaly multiplet of a supersymmetric gauge theory in a conformal supergravity background can be produced via the AdS/CFT correspondence. The type IIB supergravity in $AdS_5 \times X^5$ background reduce to a gauged supergravity in five dimensions since such a background provides a compactification on X^5 , thus the AdS/CFT correspondence implies that there should exist a holographic correspondence between the gauged supergravity in five dimensions and a four-dimensional $SU(N)$ supersymmetric gauge theory in certain phase at the large- N limit. Based on this consideration, we make use of the fact that the five-dimensional gauged supergravity admits a classical AdS_5 solution preserving the full supersymmetry. Then it is found that around this AdS_5 vacuum configuration the supermultiplet of the on-shell five-dimensional gauged supergravity converts into the off-shell conformal supergravity multiplet in four dimensions. Therefore, the holographic relation between the AdS_5 gauged supergravity and the four-dimensional supersymmetric $SU(N)$ gauge theory at the large- N limit can be established in the sense that the conformal supergravity on one hand has furnished a background for a classical superconformal gauge theory and on the other hand is a AdS_5 boundary theory for the five-dimensional gauged supergravity. Therefore, it is natural to reproduce the superconformal anomaly of a four-dimensional supersymmetric gauge theory from the AdS_5 gauged supergravity. But still it is amazing that these three distinct anomalies can be extracted in the framework of a five-dimensional gauged supergravity.

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